Hölder regularity and chaotic attractors

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We demonstrate how the Hölder regularity of a given signal is a lower bound for the Grassberger-Procaccia correlation dimension of strange attractors

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It is known from the celebrated work by Lorenz [1] that even low-dimensional deterministic dynamical systems may exhibit chaotic behavior. In the context of turbulence in fluid dynamics, Ruelle and Takens [2] have shown that the usual attractors of the asymptotic flow in the phase-space (fixed points, periodic and quasiperiodic motion) cannot explain the sensitive dependence of the solutions on the initial conditions . Attractors which display chaotic features have been called by Ruelle and Takens "strange attactors". Very often a strange attractor is a fractal object with different topological and metric (Hausdorff) dimensions. Grassberger and Procaccia [3] introduced the correlation dimension ν of strange attractors as a new measure related to the fractal dimension D by the relation $\nu < D$. The Grassberger and Procaccia method [3] is a practical algorithm to extract dimensional information from experimental data. Given a signal $\vec{X}_{i=1,N} = \vec{X}(t + \imath\tau)_{i=1,N}$, the correlation integral is defined as the standard correlation function of the time series on the attractor:

$$c(r) = \frac{1}{N^2}\sum_{i,j=1}^{N} \theta(r - |\vec{X}_i - \vec{X}_j|) \quad (N large). \tag{1}$$

where τ is the time step and $\theta(t)$ is the Heaviside function. Grassberger and Procaccia have shown that *c(r)* follows a power law of r for small r:

$$c(r) = r^{\nu} \tag{2}$$

$$\nu = \lim_{r\to 0} \frac{ln(c(r))}{ln(r)} \tag{3}$$

In an experimental situation, for very small value of r the poor statistics cause a scattering of the dimension ν, but as r increases an interval $[r_0, r_1]$ exists where the slope is constant. This region is called the scaling region. The value of the slope in the scaling region is the correlation dimension of the signal. The correlation dimension of a given signal is hence a function of the time step variable τ. Let us suppose that the signal $\vec{X}_i$ is the discretization of a continuous function $f : I \to \mathbf{R}^{\mathbf{k}}$ where $I \subset \mathbf{R}$ is an interval where $f(t + \imath\tau) = \vec{X}_i$ for $i = 1, N$. The aim of this paper is to demonstrate that, given a large interval I of fixed width $|I|$, in the limit of $N \to \infty$ and hence $\tau \to 0$, the regularity of the function f in terms of Hölder exponents gives a lower bound for the correlation dimension.
We suppose that f is continuous in terms of the Hölder exponent over the interval I:

$$f \in C^{\alpha}(I) = \left\{ f \in C(I) : \forall\, t, t' \in I\, \exists\, \alpha \in (0,1) \text{ and } \quad \exists d > 0 : |f(t) - f(t')| \le d|t - t'|^{\alpha} \right\} \tag{4}$$

then the following proposition holds:

Proposition *Let I be a large interval and $f \in C^{\alpha}(I)$ with $\alpha \in (0,1)$ and $\{\vec{X}_i\}_{i=1,N} = f(t + \imath\tau)$ with $\tau = \frac{|I|}{N}$, then in the limit of $N \to \infty$ we have $\nu \ge \frac{1}{\alpha}$*

Proof. We consider without any loss of generality the one dimensional case $k = 1$. Let $A(r)$ be the set where the Heaviside function is equal to 1

$$A(r) = \left\{ (t, t') \in I \times I : |f(t) - f(t')| < r \right\} \tag{5}$$

and

$$B(r) = \{(t,t') \in I \times I : |t-t'| < (\frac{r}{d})^{\frac{1}{\alpha}}\}. \quad (6)$$

In the discrete case the set is

$$\tilde{B}(r) = \{(i,j) \in [1,N] \times [1,N] : |i-j| < (\frac{r}{d})^{\frac{1}{\alpha}} \frac{1}{\tau}\}. \quad (7)$$

Due to relation (4), $B(r) \subseteq A(r) \quad \forall r$ and so $\mu(B(r)) \leq \mu(A(r))$, where μ denotes the usual Lebesgue measure. The cardinality of the set $B(r)$ is given by the relation $Card(B(r)) = [\sqrt{2}N(\frac{r}{d})^{\frac{1}{\alpha}}\frac{1}{\tau}]$, where the symbol $[n]$ denotes the greatest integer smaller than n. Without any loss of generality we assume $Card(B(r)) = \sqrt{2}N(\frac{r}{d})^{\frac{1}{\alpha}}\frac{1}{\tau}$. The correlation integral is then greater than the following quantity:

$$c(r) \geq \frac{\sqrt{2}}{N\tau}(\frac{r}{d})^{\frac{1}{\alpha}}. \quad (8)$$

We can now evaluate the correlation dimension of the attractor:

$$\nu \geq \lim_{r \to 0} \frac{ln(\beta r^{\frac{1}{\alpha}})}{ln(r)} \quad \beta = \frac{\sqrt{2}}{|I| d^{\frac{1}{\alpha}}}. \quad (9)$$

We cannot choose r arbitrarily small because for $(\frac{r}{d})^{\frac{1}{\alpha}}\frac{1}{\tau} < 1$ the Heaviside function is zero for every couple $(i,j) \in [1,N] \times [1,N]$. We should therefore consider the limit

$$\nu \geq \lim_{r \to (\tau)^{\alpha} d} \frac{ln(\beta r^{\frac{1}{\alpha}})}{ln(r)}. \quad (10)$$

We consider $d = 1$ without any loss of generality. Relation (10) implies that

$$\nu \geq \frac{1}{\alpha}(1 + \frac{ln(\frac{\sqrt{2}}{|I|})}{ln(\tau)}). \quad (11)$$

Given an arbitrarily small ϵ, for small enough τ and hence sufficiently large N we obtain

$$\nu \geq \frac{1}{\alpha} - \epsilon. \quad (12)$$

The significance of the stated proposition is that the regularity of a given signal provides information about the topology of the attractor. It shows that the more a given signal is non regular, in the sense of Hölder exponents, the greater is the number of dimensions required to describe the attracting set in the phase-space. We cannot have an attractor with low dimensions in the phase-space generated by a signal with strong singularities (low α). On the other hand, knowledge of the Grassberger-Procaccia correlation dimension of the attractor gives a lower bound for the regularity of the signal. An obvious example is the 2-D classical Brownian motion which we know fills the full phase-space available, i.e its attractor is the 2-D plane. From this we can infer that its Hölder exponent has to be greater than or equal to one-half, as it is [4]. Even if the propositon we have just demonstrated is quite general and applies to all non-linear dynamical systems, a natural application can be found in fluid dynamics, where we will take as the phase-space the velocity-space. In the case of the Navier-Stokes equations with the incompressibility condition and with the initial boundary conditions on $\Omega \subset \mathbf{R^d}$, the main question concerns the existence and uniqueness of the solution. The answer depends on the dimension d. In the case $d = 3$ the answer is unknown due to the possible presence of singularities of the velocity field [5,6]. From the Kolmogorov statistical theory for isotropic and homogeneous turbulence [7], we know that the velocity in the limit of $Re \to \infty$ is not smooth but Hölder continuous of exponent one-third [8]. The direct consequence of this Hölder regularity, which follows from the stated proposition, is that the fractal dimension in the three dimensional velocity-space of the asymptotic attractor of a fully developed turbulent flow must be greater than or equal to 3, as it is.

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